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A new method for Solving Semi Fully Fuzzy Linear Programming Problems

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Abstract

After successful application of fuzzy sets theory in many disciplines such as Management, Economic, Marketing and etc., this theory has been developed in the various subject as well as linear programming, fuzzy linear programming (FLP) have attracted many interests and various kinds of FLP is appeared in the literature recently. Semi-fully fuzzy linear programming problems have been investigated by many researchers in the recent decay. There are several methods to solve mentioned problems, but almost all of them need to have the basic condition for the initial solution. Hence, in this paper, we introduce a method on the primal-dual simplex algorithm which is works more efficiently than other methods by eliminating of the basic condition for the initial solution. A numerical example is given to illustrate the mentioned approach.

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1. Introduction

The successful application of the theory of fuzzy sets in control systems has led to its rapid growth in other fields such as simulation, artificial intelligence, management, operations research, and many branches of engineering sciences. One of these fields is linear programming. In many industrial and managerial problems leading to solving a linear programming problem, the decision maker cannot precisely determine the value of coefficients of a problem. In fact, in conventional linear programming, generally the coefficients of the decision problem are determined by experts with exact values, but in fuzzy environments, the assumption of accurate information by experts is far from reality. Therefore, the development and use of fuzzy modeling in real-life decision-making with inaccurate data can be appropriate. Decision making in fuzzy environments was first proposed by Bellman and Zade [1], and then the concept of fuzzy mathematical programming was proposed by Tanaka and et al. [16], in the framework of fuzzy decision-making by Bellman and Zadeh. The first formulation of the Fuzzy linear programming problem was given by Zimmerman [19]. Afterward, various models of fuzzy linear programming problems were introduced and several methods for solving it were proposed. In most of these methods, the concept of comparing fuzzy numbers is used to solve these problems. A common method for ranking fuzzy numbers is the use of ranking functions. Generally, in such a way, the fuzzy linear programming model becomes a classic linear programming model, and the solution of the main problem is determined by solving this model. This paper is organized as follows: In Section 2, we give some necessary concepts of fuzzy set theory. A review of linear programming problems with symmetric trapezoidal fuzzy numbers and the definition of the corresponding dual problem is given in Section 3. We develop and present a fuzzy primal-dual algorithm to solve the fuzzy linear programming problems in Section 4 and explain it by an illustrative example. Finally, we conclude in Section 5.

2. Preliminaries

In this section, we review the fundamental notions of fuzzy set theory (see [4] and [7]).

Definition 2.1. A convex fuzzy set $\tilde{A}$ on $\mathbb{R}$ is a fuzzy number if the following conditions hold:

- its membership function is piecewise continuous
- there exist three intervals $[a, b]$, $[b, c]$ and $[c, d]$ such that $\mu_{\tilde{A}}$ is

  increasing on $[a, b]$, equal to 1 on $[b, c]$, decreasing on $[c, d]$ and equal to 0 elsewhere

Definition 2.2. A fuzzy number on $\mathbb{R}$ (real line) is said to be a symmetric trapezoidal fuzzy number if there exist real numbers a^L and a^U, $a^u \leq a^l$ and $\rho > 0$, such that:

$$\tilde{a}(x) = \begin{cases} \frac{x}{\rho} + \frac{\rho - \rho^l}{\rho}, & x \epsilon [\, a^l - \rho, a^l] \\ 1, & x\, \epsilon [\, a^l, a^u] \\ \frac{-x}{\rho} + \frac{a^u + \rho}{\rho}, & x \epsilon [a^u, a^u + \rho] \\ 0, & otherwise \end{cases}$$

We denote a symmetric trapezoidal fuzzy number ã by
$\tilde{a} = (a^L, a^U, \alpha, \alpha)$, where $(a^l - \rho, a^u + \rho)$, is the support of $\tilde{a}$ and $[a^L, a^U]$ is its core.

Remark 2.1. We denote the set of all symmetric trapezoidal fuzzy number by $F(\mathbb{R})$.

Now, we define the arithmetic on symmetric trapezoidal fuzzy numbers. Let $\tilde{a} = (a^L, a^U, \alpha, \alpha)$ and $\tilde{b} = (b^L, b^U, \beta, \beta)$be two symmetric trapezoidal fuzzy numbers:

$$\tilde{a} + \tilde{b} = \left(a^l + b^l, a^u + b^u, \rho + \beta, \rho + \beta\right)$$

$$\tilde{a} - \tilde{b} = (a^l - b^u, a^u - b^l, \rho + \beta, \rho + \beta)$$

$$\tilde{a}\tilde{b} = \left(\left(\frac{a^l + a^u}{2}\right)\left(\frac{b^l + b^u}{2}\right) - \omega, \left(\frac{a^l + a^u}{2}\right)\left(\frac{b^l + b^u}{2}\right) + \omega, |a^u\beta + b^u\rho|, |a^u\beta + b^u\rho|\right)$$

Where

$$\theta = \min\{a^l b^l, a^l b^u, a^u b^l, a^u b^u\}, \;\; \lambda = \text{Average of}\, \theta$$

$$\delta = |\lambda - min\theta|, \;\; \gamma = |max\theta - \lambda|$$

$$\gamma = \frac{\gamma - \delta}{2},$$

From the above definition it can be seen that:

$$\lambda \geq 0, \lambda \epsilon \mathbb{R}: \qquad \lambda\tilde{a} = (\lambda a^L, \lambda a^U, \lambda\alpha, \lambda\alpha)$$

$$\lambda < 0, \lambda \epsilon \mathbb{R}: \qquad \lambda\tilde{a} = (\lambda a^L, \lambda a^U, -\lambda\alpha, -\lambda\alpha)$$

Note that depending upon the need, one can also use a smaller γ in the definition of multiplication involving symmetric trapezoidal fuzzy numbers.

Definition 2.3. Let $\tilde{a} = (a^L, a^U, \alpha, \alpha)$ and $\tilde{b} = (b^L, b^U, \beta, \beta)$ be two symmetric trapezoidal fuzzy numbers. the relations $\preccurlyeq$ and $\approx$ is defined as:

$\tilde{a} \preccurlyeq \tilde{b}$ or $\tilde{b} \succcurlyeq \tilde{a}$ if and only if :

1. $\frac{(a^l-\rho)+(a^u+\rho)}{2} < \frac{(b^l-\beta)+(b^u+\beta)}{2}$

that is $\frac{a^l+a^u}{2} < \frac{b^l+b^u}{2}$ (in this case, we may write $\tilde{a} \prec \tilde{b}$)

2. Or $\frac{a^l+a^u}{2} = \frac{b^l+b^u}{2}$, $b^l < a^l$ and $a^u < b^u$

3. Or $\frac{a^l+a^u}{2} = \frac{b^l+b^u}{2}$, $b^l = a^l$, $a^u = b^u$ and $\rho \leq \beta$.

Note that in cases (2) and (3), we also write $\tilde{a} \approx \tilde{b}$ and say that $\tilde{a}$ and $\tilde{b}$ are equivalent

Definition 2.4. For any symmetric trapezoidal fuzzy number $\tilde{a}$, we define $\tilde{a} \succcurlyeq \tilde{0}$, if there exist $\varepsilon \geq 0$ and $\rho \geq 0$ such that $\tilde{a} \succcurlyeq (-\varepsilon, \varepsilon, \rho)$. We also denote $(-\varepsilon, \varepsilon, \rho)$ by $\tilde{0}$. Note that $\tilde{0}$ is equivalent to $(0{,}0{,}0) = 0$. we consider $\tilde{0} = (0{,}0{,}0)$ as the zero symmetric trapezoidal fuzzy number.

2.2. Ranking functions

One convenient approach to solve FLP problems is based on the concept of comparison of fuzzy numbers by using ranking functions. An effective approach to order the elements of $F(\mathbb{R})$ is to define a ranking function $R\colon F(\mathbb{R}) \longrightarrow \mathbb{R}$, which maps each fuzzy number into the real line, where a natural order exists. We define orders on $F(\mathbb{R})$ by:

$\tilde{a} \succcurlyeq \tilde{b}$ if and only if $R(\tilde{a}) \geq R(\tilde{b})$

$\tilde{a} \succ \tilde{b}$ if and only if $R(\tilde{a}) > R(\tilde{b})$

$\tilde{a} \approx \tilde{b}$ if and only if $R(\tilde{a}) = R(\tilde{b})$

where ã and $\tilde{b}$ are in $F(\mathbb{R})$. Also, we write $\tilde{a} \preccurlyeq \tilde{b}$ if and only if $\tilde{b} \succcurlyeq \tilde{a}$. We restrict our attention to linear ranking functions, that is, a ranking function R such that:

$R(k\tilde{a} + \tilde{b}) = kR(\tilde{a}) + R(\tilde{b})$

for any $\tilde{a}$ and $\tilde{b}$ belonging to $F(\mathbb{R})$ and any $k \in \mathbb{R}$.

Note that $R(-\varepsilon,\varepsilon,\rho,\rho) = 0$ (we also consider $\tilde{a} \approx \tilde{0}$ if and only if $R(\tilde{a}) = \tilde{0}$).

We consider the linear ranking functions on $F(\mathbb{R})$ as:

$R(\tilde{a}) = c_L a^L + c_U a^U + c_\rho \rho + c_\rho \rho$

where $\tilde{a} = (a^L, a^U, \alpha, \alpha)$ and c_L, c_U, c_ρ are constants, at least one of which is nonzero. A special version of the linear ranking function above was first proposed by Yager [17], as follows:

$$R(\tilde{a}) = \frac{1}{2}\int_0^1 (inf\tilde{a}_\lambda + sup\tilde{a}_\lambda)d\lambda$$

which reduces to:

$$R(\tilde{a}) = \frac{a^L + a^U}{2} \tag{1}$$

Then, for symmetric trapezoidal fuzzy numbers $\tilde{a} = (a^L, a^U, \alpha, \alpha)$ and $\tilde{b} = (b^L, b^U, \beta, \beta)$, we have:

$\tilde{a} \succcurlyeq \tilde{b}$ if and only if $\frac{a^L+a^U}{2} \geq \frac{b^L+b^U}{2}$.

3. Semi fully fuzzy linear programming

Here, we review the concept of duality for such problems proposed in [7, 13, 15].

Definition 3.1. An SFFLP is defined as:

$$\begin{aligned} &min\ \tilde{z} \approx \tilde{c}\tilde{x} \\ &s.t. \begin{cases} A\tilde{x} \succcurlyeq \tilde{b} \\ \tilde{x} \succcurlyeq \tilde{0} \end{cases} \end{aligned} \tag{2}$$

where $\tilde{b}\epsilon\big(F(\mathbb{R})\big)^m$, $\tilde{c}^T\epsilon\big(F(\mathbb{R})\big)^n$, $A\epsilon\ \mathbb{R}^{m\times n}$ are given and $\tilde{x}\epsilon(F(\mathbb{R}))^n$ is to be determined.

Definition 3.2. we say that a fuzzy vector $\tilde{x}\epsilon(F(\mathbb{R}))^n$is a feasible to (1) if and only if $\tilde{x}$ satisfies the constraints and non-negativity restriction of the problem.

Definition 3.3. A feasible solution $\tilde{x}_*$ is an optimal solution for Eq. (2) if, for all feasible solutions $\tilde{x}$ for Eq. (2), we have $\tilde{c}\tilde{x}_* \preccurlyeq \tilde{c}\tilde{x}$.

Definition 3.4. Consider the $SFFLP$ problem (2) in its standard form as follows:

$$\begin{aligned} &min\ \tilde{z} \approx \tilde{c}\tilde{x} \\ &s.\mathrm{t.} \begin{cases} A\tilde{x} \approx \tilde{b} \\ \tilde{x} \succcurlyeq \tilde{0} \end{cases} \end{aligned} \tag{3}$$

where the parameters of the problem are as defined in Eq. (1).

Let $A = \left[a_{ij}\right]_{m\times n}$. Assume $rank(A) = m$. Partition A as $[B\ \ N]$, where B, $m \times m$ is nonsingular. It is obvious that $rank(B) = m$. Let y_j be the solution to $By = a_j$. It is apparent that the basic solution:

$$\tilde{x}_b = (\tilde{x}_{B_1}, \dots, \tilde{x}_{B_m})^T \approx B^{-1}\tilde{b}, \tilde{x}_N \approx \tilde{0}$$

is a solution of $A\tilde{x} = \tilde{b}$. In fact, $\tilde{x} = (\tilde{x}_B^T, \tilde{x}_N^T)^T$. If $x_B \succcurlyeq \tilde{0}$, then the fuzzy basic solution is feasible and the corresponding fuzzy objective value is $\tilde{z} \approx \tilde{c}_B\tilde{x}_B$, where $\tilde{c}_B = \left(\tilde{c}_{B_1}, \dots, \tilde{c}_{B_m}\right)$. Now, corresponding to every non-basic variable, x_j, $1 \le j \le n$, $j \ne B_i$, $i = 1, \dots, m$, define:

$$\tilde{z}_j \approx \tilde{c}_B y_j \approx \tilde{c}_B B^{-1} a_j$$

Below, we state some important results concerning to improve a feasible solution, unbounded criteria and the optimality conditions (taken from [1]).

Theorem 3.1. If we have a fuzzy basic feasible solution with a fuzzy objective value $\tilde{z}$ such that $\tilde{z}_k \succ \tilde{c}_k$ for some nonbasic variable x_k and $y_k \nleq 0$, then it is possible to obtain a new fuzzy basic feasible solution with a new fuzzy objective value $\tilde{\tilde{z}}$, that satisfies $\tilde{\tilde{z}} \preccurlyeq \tilde{z}$.

Theorem 3.2. If we have a fuzzy basic feasible solution with $\tilde{z}_k \succ \tilde{c}_k$ for some nonbasic variable k x_k and $y_k \le 0$, then the problem (Eq. (3)) has an unbounded solution.

Theorem 3.3. (Optimality conditions) If a fuzzy basic solution $\tilde{x}_B = \tilde{B}^{-1}b$, $\tilde{x}_N = \tilde{0}$ is feasible to Eq. (3) and $\tilde{z}_j \preccurlyeq \tilde{c}_j$ for all j, $1 \le j \le n$, then the fuzzy basic solution is a fuzzy optimal solution to Eq. (3).

Definition 3.5. The dual of the SFFLP problem (Eq. (2)) is defined as:

$$\begin{aligned} &max\ \tilde{u} \approx \widetilde{w}\tilde{b} \\ &s.\mathrm{t.} \begin{cases} \widetilde{w}A \preccurlyeq \tilde{c} \\ \widetilde{w} \succcurlyeq \tilde{0} \end{cases} \end{aligned} \tag{4}$$

where $\widetilde{w} = (\widetilde{w}_1, \dots, \widetilde{w}_m) \in \mathbb{R}^m$ includes the fuzzy variables corresponding to the constraints of Eq. (2). We name this problem the $DSFFLP$ problem.

Below, we give some important results concern to the $SFFLP$ problem and it's dual.

Theorem 3.4. (The weak duality property) If $\tilde{x}_0$ and $\widetilde{w}_0$ are feasible solutions to the $SFFLP$ and $DSFFLP$ problems, respectively, then $\tilde{c}\tilde{x}_0 \succcurlyeq \widetilde{w}_0\tilde{b}$.

Corollary 3.1. If $\tilde{x}_0$ and $\widetilde{w}_0$ are feasible solutions to the $SFFLP$ and $DSFFLP$ problems, respectively, and $\tilde{c}\tilde{x}_0 \approx \widetilde{w}_0\tilde{b}$, then $\tilde{x}_0$ and $\widetilde{w}_0$ are optimal solutions to their respective problems.

Corollary 3.2. If the $SFFLP$ or $DSFFLP$ problem is unbounded, then the other problem has no feasible solution.

Theorem 3.5. (Strong duality) If the $SFFLP$ or $DSFFLP$ problem has an optimal solution, then both problems have optimal solutions and the two optimal objective fuzzy values are equal (in fact, if $\tilde{x}_*$ is a fuzzy optimal solution of the primal problem, then vector $\widetilde{w}_* \approx \tilde{c}_B B^{-1}$, where B is the optimal basis, is a optimal solution of the dual problem).

Theorem 3.6. (Fundamental theorem of duality) For any $SFFLP$ problem and its corresponding $DSFFLP$ problem, exactly one of the following statements is true:

1. Both have optimal solutions $\tilde{x}_*$ and $\tilde{w}_*$ with $\tilde{c}\tilde{x}_* \succcurlyeq \tilde{w}_*\tilde{b}$
2. One problem is unbounded and the other is infeasible
3. Both problems are infeasible.

Theorem 3.7. (Complementary slackness theorem) Let $\tilde{x}_*$ and $\tilde{w}_*$ be any feasible solution to the $SFFLP$ problem and its corresponding dual problem. Then, $\tilde{x}_*$ and $\tilde{w}_*$ are optimal if and only if:

$$(\tilde{w}_* A - \tilde{c})\tilde{x}_* \approx \tilde{0}, \quad \tilde{w}_*(\tilde{b} - A\tilde{x}_*) \approx \tilde{0}$$

4. A fuzzy primal-dual method

Note that in the primal simplex method, the algorithm begins with a basic solution and in the dual simplex method, the algorithm begins with a basic optimal solution of the initial problem. In this paper, we describe a method that is called the primal-dual algorithm, which is very similar to the dual simplex method with the difference that in this algorithm, instead of starting with an initial optimal basic solution (a dual feasible basic solution) With a dual feasible solution, not necessarily basic solution, it begins to solving and with maintaining complementary slackness conditions, If there are, it achieves a dual feasible basic solution (primal optimal).
In order to explain how the mentioned algorithm works, Consider the primal and dual problems in the following standard form:

$$\begin{array}{l} min\ \tilde{z} \approx \tilde{c}\tilde{x} \\ s.t. \begin{cases} A\tilde{x} \approx \tilde{b} \\ \tilde{x} \succcurlyeq \tilde{0} \end{cases} \end{array} \quad , \quad \begin{array}{c} max\ \tilde{u} \approx \tilde{w}\tilde{b} \\ \tilde{w}A \preccurlyeq \tilde{c} \end{array} \tag{5}$$

Suppose that $\tilde{w}_0$ be the dual initial feasible solution. Our goal is to find a feasible solution to the primal problem, and consequently optimal solution to the dual problem, among the variables that holds true in the condition $\tilde{w}_0 a_j \approx \tilde{c}_j$ for all j. For this purpose, consider the set of indexes of the primal variables as follows:

$$\gamma = \{j : \tilde{w}_0 a_j - \tilde{c}_j \approx \tilde{0}\}$$

The Phase I problem which attempts to find among the indexes of the variables of set γ, a feasible solution to the primal problem, is:

$$\begin{array}{l} min \sum_{i\epsilon\gamma} \tilde{0}\tilde{x}_j + 1\tilde{x}_a \\ s.t. \begin{cases} \sum_{i\epsilon\gamma} a_j\tilde{x}_j + I\tilde{x}_a \approx \tilde{b} \\ \tilde{x}_j \succcurlyeq \tilde{0},\ j\ \epsilon\ \gamma \\ \tilde{x}_a \succcurlyeq \tilde{0} \end{cases} \end{array} \tag{6}$$

The vector of artificial variables is used to obtain the basic feasible solution to the Phase I problem. We call the Phase I problem a restricted fuzzy initial problem, because some of the initial variables can take positive values. After solving the above problem, an optimal solution is obtained for the problem. Assuming that the value of the optimal objective function of the mentioned problem is represented by $\tilde{x}^*$, if $\tilde{x}^* \approx \tilde{0}$, then a feasible basic solution can be obtained for the primal problem. In addition, the dual problem has a feasible solution and the complementary slackness condition holds. So when $\tilde{x}^* \approx \tilde{0}$, the problem has an optimal solution. If $\tilde{x}^* \succ \tilde{0}$, the obtained solution is not feasible for the primal problem, in which case we have to find another feasible solution for the dual problem in order to obtain some other new eligible variables to entering the restricted fuzzy primal problem. For this purpose, we modify the dual fuzzy vector $\tilde{w}_0$ so that:

i. Correcting $\widetilde{w}_0$ does not reduce the basic variables of the initial problem in the restricted problem
ii. At least one variable whose index was not in the γ. could be entered into the restricted problem and reduce the objective function of the $\tilde{x}^*$

To construct such a dual vector, consider the dual problem of the restricted problem:

$$\max \tilde{y}\tilde{b}$$
$$\text{s.t.}\begin{cases}\tilde{y}a_j \preccurlyeq \tilde{0}, & j \in \gamma \\ \tilde{y} \preccurlyeq \tilde{1}\end{cases} \tag{7}$$

Assuming that $\tilde{y}^*$ is the optimal solution of the above problem if $\tilde{x}_j$ is a member of the optimal base of the initial restricted problem, then obviously we should have $\tilde{y}^*a_j \approx \tilde{0}$. The entry condition to the base of the initial restricted problem is the satisfaction of the corresponding dual constraint in relation $\tilde{y}^*a_j \succ \tilde{0}$. However, no variables in the initial restricted problem are this property because the problem has been solved and the conditions for optimization have been found. For all $j \notin \gamma$, we calculate $\tilde{y}^*a_j$. If $\tilde{y}^*a_j \succ \tilde{0}$, then $\tilde{x}_j$ can be entered into the restricted problem with the aim of reducing $\tilde{x}^*$. So, we are looking for a way to enter $\tilde{x}_j$ into the restricted problem with this Property.

We construct the dual vector $\widetilde{w}_0'$with $\rho > 0$ as follows:

$$\widetilde{w}_0' \approx \widetilde{w}_0 + \rho\tilde{y}^*$$

Then:

$$\widetilde{w}_0'a_j - \tilde{c}_j \approx (\widetilde{w}_0 + \rho\tilde{y}^*)a_j - \tilde{c}_j \approx \left(\widetilde{w}_0a_j - \tilde{c}_j\right) + \rho\left(\tilde{y}^*a_j\right) \tag{8}$$

Notice that for each $j \in \gamma$ we have:

$$\widetilde{w}_0a_j - \tilde{c}_j \approx \tilde{0}, \quad \tilde{y}^*a_j \preccurlyeq \tilde{0}$$

Therefore, for each $j \in \gamma$, Eq. (8) implies that:

$$\widetilde{w}_0'a_j - \tilde{c}_j \preccurlyeq \tilde{0}$$

Also, if $j \in \gamma$, $\tilde{x}_j$ be the basic variable of the problem, we will have:

$$\widetilde{w}_0'a_j - \tilde{c}_j \approx \tilde{0}, \quad \tilde{y}^*a_j \approx \tilde{0}$$

So $\tilde{x}_j$ in this situation can remain in the problem because of $\widetilde{w}_0'\mathrm{a}_j - \tilde{c}_j \preccurlyeq \tilde{0}$. If $j \notin \gamma$, $\tilde{y}^*a_j \preccurlyeq \tilde{0}$ then:

$$\widetilde{w}_0'a_j - \tilde{c}_j \approx \left(\widetilde{w}_0a_j - \tilde{c}_j + \rho\tilde{y}^*a_j\right) \preccurlyeq \tilde{0}$$

If $j \notin \gamma$, $\tilde{y}^*a_j \succ \tilde{0}$ The ρ should be chosen in such a way that $\widetilde{w}_0'a_j - \tilde{c}_j \preccurlyeq \tilde{0}$. Notice that for each j, $\widetilde{w}_0'a_j - \tilde{c}_j \preccurlyeq \tilde{0}$ and for each $j \notin \gamma$, $\widetilde{w}_0a_j - \tilde{c}_j \prec \tilde{0}$ because if $\widetilde{w}_0a_j - \tilde{c}_j \approx \tilde{0}$, this index was exist in γ. So for each $j \notin \gamma$ and $\tilde{y}^*a_j \succ \tilde{0}$ we must have:

$$\widetilde{w}_0'a_j - \tilde{c}_j \preccurlyeq \tilde{0}$$
$$\left(\widetilde{w}_0a_j - \tilde{c}_j\right) + \rho\left(\tilde{y}^*a_j\right) \preccurlyeq \tilde{0}$$
$$\rho\tilde{y}^*a_j \preccurlyeq -\left(\widetilde{w}_0a_j - \tilde{c}_j\right)$$
$$\rho = \min_j\left\{\frac{-(\widetilde{w}_0a_j - \tilde{c}_j)}{\tilde{y}^*a_j}\,\middle|\,\tilde{y}^*a_j \succ \tilde{0}\right\} \tag{9}$$

From Eq. (9) at least for one $j \notin \gamma$ we have:

$$\widetilde{w}_0'a_j - \tilde{c}_j \approx \tilde{0}$$

If we call this index k, we will have:

$$\widetilde{w}_0'a_k - \tilde{c}_k \approx \tilde{0}$$

In addition, $\tilde{\hat{v}}a_k \succ \tilde{0}$. So, for each $j = 1, \ldots, n$ we obtain:

$$\widetilde{w}_0'a_j - \tilde{c}_j \preccurlyeq \tilde{0}$$

So $\widetilde{w}_0'$ is the dual feasible solution. Thus $\tilde{x}_k$ enters into the initial restricted problem.

4.1 dual infinite solution situation

Previous Entries continue until $\tilde{x}^* \approx \tilde{0}$, which In this case, a dual feasible basic solution will be obtained and or we conclude that $\tilde{x}^* \succ \tilde{0}$ and for each $j \notin \gamma$ *we have* $\tilde{y}^* a_j \preccurlyeq \tilde{0}$. Consider in this case:

$$\widetilde{w}_0' \approx \widetilde{w}_0 + \rho \tilde{y}^*$$

Since for each j, $\widetilde{w}_0' a_j - \tilde{c}_j \preccurlyeq \tilde{0}$ and $\tilde{y}^* a_j \preccurlyeq \tilde{0}$, then for each j and $\rho > 0$, we will have:

$$\widetilde{w}_0' a_j - \tilde{c}_j \approx \left(\widetilde{w}_0 a_j - \tilde{c}_j\right) + \rho\left(\tilde{y}^* a_j\right) \preccurlyeq \tilde{0}$$

That is $\widetilde{w}_0'$ for each $\rho > 0$ is feasible for dual, and Furthermore:

$$\widetilde{w}_0' \tilde{b} \approx (\widetilde{w}_0 + \rho \tilde{y}^*)\tilde{b} \approx \widetilde{w}_0 \hat{b} + \rho \tilde{y}^* \tilde{b}$$

According to the strong duality theorem $\tilde{y}^* \tilde{b} \approx \tilde{x}^* \succ \tilde{0}$, then $\widetilde{w}_0' \tilde{b}$ increases with increasing ρ, and if $\rho \longrightarrow +\infty$ then $\widetilde{w}_0' \tilde{b} \longrightarrow +\infty$. That is, the dual problem has an infinite answer, and the Initial problem is infeasible.

The fuzzy dual-primal simplex algorithm (for minimization)

Initial step

Choose a fuzzy vector $\widetilde{w}_0$ so that for each j:

$$\widetilde{w}_0 a_j - \tilde{c}_j \preccurlyeq \tilde{0}$$

Main step

1. Suppose:

$$\gamma = \{j : \widetilde{w}_0 a_j - \tilde{c}_j \approx \tilde{0}\}$$

   Solve the initial restricted problem below.

$$min \sum_{i \epsilon \gamma} \tilde{0}\tilde{x}_j + 1\tilde{x}_a$$

$$s.t. \begin{cases} \sum_{i \epsilon \gamma} a_j \tilde{x}_j + I\tilde{x}_a \approx \tilde{b} \\ \tilde{x}_j \succcurlyeq \tilde{0},\ j \epsilon \gamma \\ \tilde{x}_a \succcurlyeq \tilde{0} \end{cases}$$

   Let $\tilde{x}^*$ be the optimal solution to the objective function.
   If $\tilde{x}^* \approx \tilde{0}$, then stop (The solution obtained is an optimal solution to the problem).
   If $\tilde{x}^* \succ \tilde{0}$, let $\tilde{y}^*$ be the dual optimal solution to the above restricted problem.
2. If for each j, $\tilde{y}^* a_j \preccurlyeq \tilde{0}$ then stop. The dual problem is infinite and therefore, the initial problem is infeasible.

Else let:

$$\rho = \min_j \left\{ \frac{-\left(\widetilde{w}_0 a_j - \tilde{c}_j\right)}{\tilde{y}^* a_j} \middle| \tilde{y}^* a_j \succ \tilde{0} \right\}$$

Replace $\widetilde{w}_0$ with $\widetilde{w}_0 + \rho \tilde{y}^*$ (which is a dual feasible solution) and go back to step 1. Check the following example.

Example 4.1. Consider the following FLP problem:

$$min \quad \tilde{z} \approx (1,5,1,1)\tilde{x}_1 + (2,6,1,1)\tilde{x}_2 + (5,7,2,2)\tilde{x}_3 + (6,8,1,1)\tilde{x}_4 - (0,2,1,1)\tilde{x}_5$$

$$s.t. \begin{cases} 2\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + 6\tilde{x}_4 - 5\tilde{x}_5 \succcurlyeq (6,10,2,2) \\ \tilde{x}_1 + \tilde{x}_2 + 2\tilde{x}_3 + \tilde{x}_4 + 2\tilde{x}_5 \succcurlyeq (1,5,1,2) \\ \tilde{x}_1, \dots, \tilde{x}_5 \succcurlyeq \tilde{0} \end{cases}$$

The dual of the above problem is as follow:

$$max\ \tilde{w} \approx (6,10,2,2)\tilde{w}_1 + (1,5,1,1)\tilde{w}_2$$

$$s.t. \begin{cases} 2\,\tilde{w}_1 + \tilde{w}_2 \preccurlyeq (1,5,1,1) \\ \tilde{w}_1 + \tilde{w}_2 \preccurlyeq (2,6,2,2) \\ \tilde{w}_1 + 2\tilde{w}_2 \preccurlyeq (5,7,2,2) \\ 6\,\tilde{w}_1 + \tilde{w}_2 \preccurlyeq (6,8,1,1) \\ -5\,\tilde{w}_1 + 2\tilde{w}_2 \preccurlyeq (0,\text{-}2,1,1) \\ -\tilde{w}_1 \preccurlyeq \tilde{0} \\ -\tilde{w}_2 \preccurlyeq \tilde{0} \\ \tilde{w}_2 \text{ و } \tilde{w}_1 \quad unrestricted \end{cases}$$

A feasibale solution to the dual problem is $\tilde{w}_0 \approx (0,0)$. By replacing these values in dual constraints, $\gamma = \{6, 7\}$ is obtained. If $\tilde{x}_8$ and $\tilde{x}_9$ are considered as artificial variables, the restricted problem will be obtained as follows:

$$min(1,1,0,0)\tilde{x}_8 + (1,1,0,0)\tilde{x}_9$$

$$\text{s.t.} \begin{cases} \text{-}\tilde{x}_6 + \tilde{x}_8 \approx (6,10,2,2) \\ \text{-}\tilde{x}_7 + \tilde{x}_9 \approx (1,5,1,1) \\ \tilde{x}_6,\tilde{x}_7,\tilde{x}_8,\tilde{x}_9 \succcurlyeq \tilde{0} \end{cases}$$

The optimal solution to the above problem based on Yager ranking function is:

$$R(\tilde{x}_6) = 0,\ R(\tilde{x}_7) = 0,\ R(\tilde{x}_8) = 8,\ R(\tilde{x}_9) = 3,\ R(\tilde{x}^0) = 11$$

And from the complementary slackness $\tilde{y}^* \approx ({\tilde{y}^*}_1,{\tilde{y}^*}_2) \approx ((1,1,0,0),(1,1,0,0))$.

Then, the parameter of ρ is determined as follows:

$$\rho = min\left\{\frac{\text{-(-2)}}{\frac{1}{2}}\right\} = 4$$

and then replace $\tilde{w}_0$ by $\tilde{w}_0 + \rho\tilde{y}^*$ as follows:

$((0,0,0,0),(0,0,0,0))+1((1,1,0,0),(1,1,0,0))=((1,1,0,0),(1,1,0,0))$.

Now with continuing this process, the optimal solution to the main problem based on Yager ranking function is:

$$R(\tilde{x}_2) = R(\tilde{x}_3) = R(\tilde{x}_5) = 0,\ R(\tilde{x}_1) = 2.5,\ R(\tilde{x}_4) = 0.5$$

4.2 Finite convergence of the primal-dual method

We recall that in each iteration, a Modified variable is added to the initial restricted problem. Therefore, in absence of degeneracy in the initial restricted problem, the objective function $\tilde{x}^*$ is strictly descending in each iteration. This means that the set Γ that generated in each iteration is distinct of all those generated in the previous iteration. Because there is only a finite number of sets in the form γ, (Remember that $\gamma \subset \{1,2, \dots, n\}$), and none of them can be repeated, So the algorithm ends in a finite number of steps. Note that the primal-dual method essentially works to minimize the sum of the artificial variables $\tilde{x}_a$ of the restricted problem with attention to $A\tilde{x} + I\tilde{x}_a \approx \tilde{b}, \tilde{x} \succcurlyeq \tilde{0}, \tilde{x}_a \succcurlyeq \tilde{0}$. However, through the vector $\tilde{w}_0$, we can select nonbasic variables to enter the base. At each step, or for all j, $\tilde{z}_j - \tilde{c}_j = \tilde{y}a_j \preccurlyeq \tilde{0}$, that in which case the restricted problem is solved, or for some nonbasic variables, $\tilde{z}_j - \tilde{c}_j = \tilde{y}a_j \succ \tilde{0}$. In the previous case, if $\tilde{x}^* \succ \tilde{0}$, the primal problem is infeasible because the restricted problem has an optimal positive solution. On the other hand, if $\tilde{x}^* \approx \tilde{0}$, the primal-dual problem is optimal. Otherwise, If the restricted problem has not been resolved so far, the calculation of $\tilde{w}_0$ in each iteration guarantees that a nonbasic variable such $\tilde{x}_j$ with $\tilde{z}_j - \tilde{c}_j = \tilde{y}a_j \succ \tilde{0}$ is presented in set γ and hence is allowed to enter. So, we are applying simplex algorithm of the primal program on the initial restricted problem with the constraint on input variable criteria. Consequently, we can obtain finite convergence even in degeneracy by using the lexicographic Rule which is independent of how the input variable is selected.

5. Conclusions

In this paper, we presented a new method based on the primal-dual algorithm to solve SFFLP problems. The proposed method, for solving problems SFFLP can be more efficient than other methods if there is an initial feasible solution. For this reason, the development of the mentioned method in the other environments such as network optimization and data envelopment analysis can be very useful.

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